Quantum Mechanics I

Week 9 (Solutions)

Spring Semester 2025

1 The Quantum Mechanical Virial Theorem

Consider a particle in one dimension whose Hamiltonian is given by

$$H = \frac{p^2}{2m} + V(x). \tag{1.1}$$

(a) Show that the time evolution of the expectation value of an observable is

$$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t}\right\rangle. \tag{1.2}$$

Show this in the Schrödinger picture.

Let us consider the time derivative of the expectation value of some observable, Q and expand as follows:

$$\frac{d}{dt}\langle Q\rangle = \frac{d}{dt}\langle\Psi|\hat{Q}|\Psi\rangle = \left\langle \frac{\partial\Psi}{\partial t}\middle|\hat{Q}\middle|\Psi\right\rangle + \left\langle \Psi\middle|\frac{\partial\hat{Q}}{\partial t}\middle|\Psi\right\rangle + \left\langle \Psi\middle|\hat{Q}\middle|\frac{\partial\Psi}{\partial t}\right\rangle.$$

Now, the Schrödinger equation says

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$$

(where $H = \frac{p^2}{2m} + V$ is the Hamiltonian). So

$$\frac{d}{dt}\langle Q\rangle = -\frac{1}{i\hbar}\langle\hat{H}\Psi|\hat{Q}|\Psi\rangle + \frac{1}{i\hbar}\langle\Psi|\hat{Q}\hat{H}|\Psi\rangle + \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle.$$

But $\hat{H}$ is Hermitian, so $\langle\hat{H}\Psi|\hat{Q}|\Psi\rangle = \langle\Psi|\hat{H}\hat{Q}|\Psi\rangle$, and hence

$$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle.$$

(b) Show the following

$$\frac{d}{dt}\langle xp\rangle = \left\langle \frac{p^2}{m} \right\rangle - \left\langle x\frac{\mathrm{d}V}{\mathrm{d}x} \right\rangle. \tag{1.3}$$

The operator $Q = xp$ does not have any explicit time dependence, thus the second term in the RHS Eq. (**??**) is zero. The first term of the same equation contains a commutator, which we can easily tackle:

$$\begin{aligned}[H, xp] &= [H, x]p + x[H, p] \\ &= -\frac{i\hbar p}{m}\cdot p + x\cdot i\hbar\frac{\partial V}{\partial x} \\ &= -i\hbar\Big[\frac{p^2}{m} - x\frac{\partial V}{\partial x}\Big].\end{aligned}$$

We then use this result into Eq. (**??**), and thus find:

$$\begin{aligned}\frac{d}{dt}\langle xp\rangle &= \frac{i}{\hbar}\left[-\frac{i\hbar}{m}\langle p^2\rangle + i\hbar\langle x\frac{\partial V}{\partial x}\rangle\right] \\ &= 2\left\langle \frac{p^2}{2m} \right\rangle - \left\langle x\frac{\partial V}{\partial x} \right\rangle.\end{aligned}$$

(c) The classical virial theorem relates the time average of the total kinetic energy T of a system (bound by some conservative force) with the "virial" of the system, and is expressed as follows:

$$\langle T\rangle = -\frac{1}{2}\langle Fx\rangle \ . \tag{1.4}$$

Here F is the force and the right-hand side $-\frac{1}{2}\langle Fx\rangle$, proportional to the average of the product Fx is called the "virial". To identify the preceding relation with the quantum-mechanical analogue of the virial theorem it is essential that the left-hand side of Eq. (**??**) vanishes. Under what condition would this happen?

The LHS of Eq. (**??**) becomes zero when the state we consider is a stationary state. In a stationary state, all expectation values with any (time-independent) operator do not depend on time. Thus $d\langle xp\rangle/dt = 0$. In this case, Eq. (**??**) becomes:

$$\left\langle \frac{p^2}{2m} \right\rangle = \frac{1}{2}\left\langle x\frac{\partial V}{\partial x} \right\rangle.$$

Using the definition of the force in terms of the potential energy $F = -\partial V/\partial x$, we find:

$$\left\langle \frac{p^2}{2m} \right\rangle = -\frac{1}{2}\langle Fx\rangle,$$

which is the quantum-mechanical analogue of the classical virial theorem. In the previous exercise set, we show this relation for the stationary states of the harmonic oscillator. For the same system, we can also show, using the virial theorem, that the expectation values of the kinetic and potential energies are equal in a stationary state. Using the potential of the harmonic oscillator $V = \frac{1}{2}m\omega^2 x^2$, we find:

$$x\frac{dV}{dx} = m\omega^2 x^2 = 2V$$

thus $\langle T\rangle = \langle V\rangle$.

A similar result can be obtained for a potential of the form $V(x) = Ax^{2n}$. In this case $Fx = -x\mathrm{d}V/\mathrm{d}x = -2nV$ so the virial theorem can be expressed as $\langle T\rangle = n\langle V\rangle$.

The virial theorem can be easily generalized in three dimensions.

2 The Unstable Particle

We would like to describe an unstable particle that spontaneously decays with a lifetime τ. In this case, the total probability of finding the particle somewhere would not be constant, but rather should decrease at an exponential rate:

$$P(t) = \int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = e^{-t/\tau} \tag{2.1}$$

One way of achieving this result is to assume a complex potential $V(\mathbf{r})$. In class, we have dealt only with real potential, for which we showed that the probability is conserved. In our example, consider a complex potential in the form of:

$$V = V_0 - i\Gamma \tag{2.2}$$

where V_0 is the true potential energy of the system and $\Gamma > 0$ is a real constant.

(a) Is the Hamiltonian Hermitian?

With a complex potential, the Hamiltonian is no longer a Hermitian operator. The Hamiltonian reads:

$$H = \frac{p^2}{2m} + V_0 - i\Gamma.$$

We look at the matrix elements in the position representation where we have:

$$\langle x|H|y\rangle = \frac{1}{2m}\langle x|p^2|y\rangle + V_0\delta(x-y) - i\Gamma\delta(x-y).$$

From the matrix representation of the Hamiltonian operator, we observe that in this particular case, the diagonal elements are complex. As a result, the Hamiltonian is identified as a non-Hermitian operator.

(b) Show that that the rate of total probability is

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -\frac{2\Gamma}{\hbar}P. \tag{2.3}$$

We consider the rate of change of the spatial integral of the probability density as follows

$$\frac{d}{dt}\int_{-\infty}^{+\infty}|\Psi(x,t)|^2\,dx \;=\; \int_{-\infty}^{+\infty}\frac{\partial}{\partial t}\left|\Psi(x,t)\right|^2 dx,$$

where we have exchanged the time derivative and the spatial integral. Notice that the total derivative is replaced by a partial one. We isolate now the integrand,

$$\frac{\partial}{\partial t}\left|\Psi\right|^2 = \frac{\partial}{\partial t}\big(\Psi^*\,\Psi\big) = \Psi^*\,\frac{\partial\Psi}{\partial t} + \frac{\partial\Psi^*}{\partial t}\,\Psi.$$

The time derivative of the wavefunction is given by the Schrödinger equation

$$\frac{\partial\Psi}{\partial t} = -\frac{i}{\hbar}\left[-\frac{\hbar^2}{2m}\,\frac{\partial^2\Psi}{\partial x^2} + V_0\Psi - i\Gamma\Psi\right], \qquad \frac{\partial\Psi^*}{\partial t} = \frac{i}{\hbar}\left[-\frac{\hbar^2}{2m}\,\frac{\partial^2\Psi^*}{\partial x^2} + V_0\Psi^* + i\Gamma\Psi^*\right].$$

Then by introducing the latter equations to the integrand, we find:

$$\begin{aligned}\frac{\partial}{\partial t}\,|\Psi|^2 &= \frac{i\hbar}{2m}\left(\Psi^*\,\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi^*}{\partial x^2}\,\Psi\right) - \frac{2\Gamma}{\hbar}|\Psi|^2\\ &= \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^*\,\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\,\Psi\right)\right] - \frac{2\Gamma}{\hbar}|\Psi|^2.\end{aligned}$$

We integrate this result over the whole space,

$$\frac{d}{dt}\int_{-\infty}^{+\infty}|\Psi(x,t)|^2\,dx = \frac{i\,\hbar}{2\,m}\left[\Psi^*\,\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\,\Psi\right]_{x=-\infty}^{x=+\infty} - \frac{2\Gamma}{\hbar}\int_{-\infty}^{+\infty}|\Psi(x,t)|^2\,dx.$$

The first term yields zero since the wavefunction goes to zero at $x \to \pm\infty$ (since it is assumed to be normalizable). Thus we find:

$$\frac{d}{dt}P(t) = -\frac{2\Gamma}{\hbar}P(t), \quad P(t) = \int_{-\infty}^{+\infty}|\Psi(x,t)|^2\,dx,$$

which is the desired result. Therefore, the imaginary part of the potential describes some loss mechanism, and thus probability is no longer conserved. If the potential was real, i.e. $\Gamma = 0$, then the RHS of the rate equation becomes zero and the probability is conserved, as expected.

(c) Solve for $P(t)$ and find the lifetime of the particle in terms of Γ.

The rate equation we found in the previous question is a simple first-order ordinary differential equation and can be trivially solved to find

$$P(t) = e^{-2\Gamma/\hbar}. \tag{2.4}$$

where we have used $P(0) = 1$. From this solution, we can extract the lifetime of the particle $\tau = \hbar/2\Gamma$.

3 Overlapping Wavefunctions

Consider a particle and two normalized energy eigenfunctions $\psi_1(x)$ and $\psi_2(x)$ corresponding to the eigenvalues $E_1 \neq E_2$. Assume that the eigenfunctions vanish outside the two non-overlapping regions Ω_1 and Ω_2 respectively.

(a) Show that, if the particle is initially in region Ω_1 then it will stay there forever.

Clearly $\psi(\boldsymbol{x}, t) = e^{-iEt/\hbar}\,\psi_1(\boldsymbol{x})$ implies that $|\psi(\boldsymbol{x}, t)|^2 = |\psi_1(\boldsymbol{x})|^2$, which vanishes outside Ω_1 at all times.

(b) If, initially, the particle is in the state with wave function

$$\psi(\boldsymbol{x}, 0) \;=\; \frac{1}{\sqrt{2}}\big[\psi_1(\boldsymbol{x}) + \psi_2(\boldsymbol{x})\big],$$

show that the probability density $|\psi(\boldsymbol{x}, t)|^2$ is independent of time.

If the two regions do not overlap, then $\psi_1(\boldsymbol{x})\,\psi_2^*(\boldsymbol{x}) = 0$ everywhere, and hence

$$|\psi(\boldsymbol{x}, t)|^2 = \tfrac{1}{2}\left[\,|\psi_1(\boldsymbol{x})|^2 + |\psi_2(\boldsymbol{x})|^2\right], \tag{3.1}$$

which is time independent.

(c) Now assume that the two regions Ω_1 and Ω_2 overlap partially. Starting with the initial wave function of case (b), show that the probability density is a periodic function of time.

If the two regions overlap, the probability density becomes

$$|\psi(\boldsymbol{x}, t)|^2 = \frac{1}{2}\Big[\big|\psi_1(\boldsymbol{x})\big|^2 + \big|\psi_2(\boldsymbol{x})\big|^2\Big] + \big|\psi_1(\boldsymbol{x})\big|\,\big|\psi_2(\boldsymbol{x})\big|\,\cos\big[\phi_1(\boldsymbol{x}) - \phi_2(\boldsymbol{x}) - \omega\,t\big]. \tag{3.2}$$

Here we set $\psi_{1,2} = \big|\psi_{1,2}\big|\exp\big[i\,\phi_{1,2}\big]$ and $E_1 - E_2 = \hbar\,\omega$. This is clearly a periodic function of time with period $T = 2\pi/\omega$.

(d) Starting with the same initial wave function and assuming that the two eigenfunctions are real and isotropic, take the two partially overlapping regions Ω_1 and Ω_2 to be two concentric spheres of radii $R_1 > R_2$. Compute the probability current that flows through Ω_1.

The current density is readily computed:

$$\mathbf{J} = \hat{\mathbf{r}}\,\frac{\hbar}{2\,m}\,\sin(\omega\,t)\Big[\psi_2'(r)\,\psi_1(r) \;-\; \psi_1'(r)\,\psi_2(r)\Big], \tag{3.3}$$

and vanishes at $r = R_1$, since one or the other eigenfunction vanishes there. This can be seen through the continuity equation in the following alternative way:

$$\begin{aligned}
I_{\Omega_1} &= -\frac{\mathrm{d}}{\mathrm{d}t} P_{\Omega_1} \\
&= \int_{S(\Omega_1)} d\mathbf{S} \cdot \mathbf{J} \\
&= \int_{\Omega_1} d^3x \, \nabla \cdot \mathbf{J} \\
&= -\int_{\Omega_1} d^3x \, \frac{\partial}{\partial t} \left|\psi(\boldsymbol{x}, t)\right|^2 \\
&= \omega \, \sin(\omega \, t) \int_{\Omega_1} d^3x \, \psi_1(r) \, \psi_2(r).
\end{aligned}$$

The last integral vanishes because the two eigenfunctions are orthogonal.

Hint: Use the polar form for a generic wavefunctions $\psi_i(\boldsymbol{x})$, i.e.

$$\psi_i(\boldsymbol{x}) = |\psi_i(\boldsymbol{x})| e^{i\phi_i(\boldsymbol{x})} \, .$$

4 Probability Current in One Dimension

A. Consider a particle moving in one dimension. Suppose that at time $t = 0$ the quantum state of the particle is characterized by a wave function $\psi(x, t = 0) = \psi_0(x)$ which is real-valued ($\psi_0(x) = \psi_0^*(x)$).

(a) Show that at $t = 0$ the particle has a vanishing average momentum, i.e. $\langle p \rangle_{\psi_0} = 0$. What is the probability current density J at $t = 0$? Do you expect that the current will change in time at $t > 0$?

The probability current density at time $t = 0$ vanishes:

$$J = \frac{\hbar}{2m \, i} \left[\psi_0^* \, \psi_0' - \psi_0 \, (\psi_0^*)'\right] = 0. \tag{4.1}$$

The expectation value of the momentum in this state is also zero:

$$\begin{aligned}
\langle \psi_0 | p | \psi_0 \rangle &= -i\hbar \int_{-\infty}^{\infty} \psi_0(x) \, \psi_0'(x) \, dx \\
&= -\frac{i\hbar}{2} \int_{-\infty}^{\infty} \frac{d}{dx} \left[\psi_0^2(x)\right] dx \\
&= -\frac{i\hbar}{2} \left[\psi_0^2(x)\right]_{-\infty}^{\infty} \\
&= 0.
\end{aligned}$$

This is consistent with the general relation between the average momentum and the integral of the probability current, that was demonstrated in the lecture notes:

$$\langle \psi_0 | \hat{p} | \psi_0 \rangle = m \int_{-\infty}^{\infty} dx J(x) \; . \tag{4.2}$$

The current is $J(x) = 0$ at time $t = 0$. At later times, the probability current will change if ψ_0 is not an eigenstate of the Hamiltonian. In this case, the wavefunction $\psi(x, t)$ at $t > 0$ cannot be assumed to be real valued. If instead ψ_0 is a stationary state of the Hamiltonian, the wavefunction and, thus, the current will be constant and $J = 0$ at all times. Thus, the answer does not only depend on the function ψ_0 but also on the Hamiltonian which defines the dynamics of the system.

(b) Suppose instead that at $t = 0$ the state is instead $\chi_0(x) = e^{ikx}\psi_0(x)$ with $\psi_0(x)$ real valued. Show that this state has average momentum $p_0 = \hbar k$.

Consider the wave function

$$\chi_0(x) = e^{\,i\,p_0 x/\hbar}\,\psi_0(x). \tag{4.3}$$

Its momentum expectation value is found by evaluating

$$\langle\chi_0|\hat{p}|\chi_0\rangle = -\,i\hbar\int_{-\infty}^{\infty} e^{-\,i\,p_0 x/\hbar}\,\psi_0(x)\,\left[e^{\,i\,p_0 x/\hbar}\,\psi_0(x)\right]'\,dx. \tag{4.4}$$

Expanding the derivative in the integrand gives

$$-\,i\hbar\int_{-\infty}^{\infty}\psi_0(x)\left(\tfrac{i\,p_0}{\hbar}\,\psi_0(x) + \psi_0'(x)\right)dx = p_0 + \langle p\rangle_{\psi_0} = p_0, \tag{4.5}$$

since we have already established $\langle p\rangle_{\psi_0} = 0$ in this case. The inclusion of this specific phase factor (plane wave with momentum p_0) results in a non-zero momentum as expected.

(c) Show that the corresponding momentum wave function $\tilde{\chi}_0(p)$ is translated in momentum, i.e. $\tilde{\chi}(p) = \tilde{\psi}\,(p - p_0)$.

The momentum wave function is

$$\tilde{\chi}_0(p) = \int\frac{dx}{\sqrt{2\pi\,\hbar}}\,e^{-\frac{i\,p\,x}{\hbar}}\,\chi_0(x) = \int\frac{dx}{\sqrt{2\pi\,\hbar}}\,e^{\frac{-i\,(p-p_0)\,x}{\hbar}}\,\psi_0(x) = \tilde{\psi}_0\big(p - p_0\big). \tag{4.6}$$

B. Calculate the probability current $J(x)$ for the following wavefunctions:

We will use the following relation for the probability current in one dimension:

$$J = \frac{\hbar}{2m\,i}\left[\psi^*\frac{\mathrm{d}\psi}{\mathrm{d}x} - c.c.\right] = \frac{\hbar}{2m\,i}2iIm\left\{\psi^*\frac{\mathrm{d}\psi}{\mathrm{d}x}\right\} = \frac{\hbar}{m}Im\left\{\psi^*\frac{\mathrm{d}\psi}{\mathrm{d}x}\right\}.$$

(a) $\psi(x) = A\,e^{-\gamma x}$, where A is a complex constant and γ is a real constant.

This wavefunction represents a localized state. We first compute:

$$\begin{aligned}\psi^*\frac{\mathrm{d}\psi}{\mathrm{d}x} &= \left[A^* e^{-\gamma x}\right]\cdot\left[-\gamma A e^{-\gamma x}\right]\\ &= -\gamma|A|^2 e^{-2\gamma x}.\end{aligned}$$

This quantity is real, thus $Im\{\psi^*\,d\psi/dx\}$ and as a further consequence, the probability current will be zero. We expect this result since the wavefunction is real and corresponds to a bound state.

(b) $\psi(x) = N(x)\, e^{i\,S(x)/\hbar}$, where $N(x)$ and $S(x)$ are real.

This wavefunction is a generalized polar form representation, where both the amplitude and phase can depend on the position. We first compute:

$$\begin{aligned}\psi^* \frac{\mathrm{d}\psi}{\mathrm{d}x} &= \left[N(x)e^{-iS(x)/\hbar}\right] \cdot \left[\frac{\mathrm{d}N(x)}{\mathrm{d}x}e^{iS(x)/\hbar} + N(x)\frac{i}{\hbar}\frac{\mathrm{d}S(x)}{\mathrm{d}x}e^{iS(x)/\hbar}\right] \\ &= N(x)\frac{\mathrm{d}N(x)}{\mathrm{d}x} + \frac{i}{\hbar}N^2(x)\frac{\mathrm{d}S(x)}{\mathrm{d}x}\end{aligned}$$

and thus the probability current is

$$J(x) = \frac{1}{m}N^2(x)\frac{\mathrm{d}S(x)}{\mathrm{d}x}.$$

The probability current will depend on the phase of the wavefunction.

(c) $\psi(x) = A\,e^{ikx} + B\,e^{-ikx}$, where A, B, k are real constants. This wavefunction represents two waves propagating in opposite directions with the same wavenumber, with different probability amplitudes. We first compute:

$$\begin{aligned}\psi^* \frac{\mathrm{d}\psi}{\mathrm{d}x} &= ik\left[Ae^{-ikx} + Be^{ikx}\right] \cdot \left[Ae^{ikx} - Be^{-ikx}\right] \\ &= ik\left[|A|^2 - |B|^2 + i2AB\sin 2kx\right] \\ &= -2kAB\sin 2kx + ik\left[|A|^2 - |B|^2\right].\end{aligned}$$

Then the current is found:

$$J(x) = \frac{\hbar k}{m}\left[|A|^2 - |B|^2\right].$$

The probability current flows depending on the amplitudes of the two waves. For equal amplitudes, the current is zero.

C. Consider a wavefunction $\psi(x)$, not necessarily real-valued and not necessarily localized (normalizable). The only thing which we know is that $\psi(x)$ is an eigenstate of the Hamiltonian $\hat{H} = \hat{p}^2/2m + V(\hat{x})$ for a certain potential $V(\hat{x})$. (The motion is assumed to be as before one-dimensional). What can you say about the current $J(x)$? What changes in bound states as compared to scattering states?

In any stationary state, the probability density $|\psi(x)|^2$ is constant in time. In one dimension, this implies that the current must be constant: $J(x) = J$ (independent of x). (In three dimensions, this would not be necessarily true; one could find currents which are not constant but which still do not lead to a time-variation of the probability). In a bound state, the current J must, in addition, be 0. Otherwise, we would have a net flow of probability, which is inconsistent with a localized state. However, in general there may also be states with $J \neq 0$. These correspond to scattering states, which exist if the potential $V(x)$ approaches a finite value at $x \to \pm\infty$.

5 Galilean invariance of the Schrödinger equation

We want to show that the Schrödinger equation is invariant under the Galilean transformation. Consider a particle in a frame K' described by

$$i\hbar\frac{\partial\Psi'(\mathbf{r}',t')}{\partial t'} = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}'^2\Psi'(\mathbf{r}',t') + V'(\mathbf{r}',t')\Psi'(\mathbf{r}',t'), \tag{5.1}$$

where $V'(r',t') = V(r,t)$. We write the equation in a new K frame that moves with speed v with respect to K' as

$$\mathbf{r}' = \mathbf{r} - \boldsymbol{v}t, \quad t' = t. \tag{5.2}$$

(a) Find the expression of the derivatives in the frame K' in terms of those in the frame K.

The easiest procedure is to perform the calculation in one dimension, to later generalize the result to the 3D case. In 1D, the transformations are:

$$x = x' + v_x t', \quad t = t'.$$

The derivatives can be expressed with respect to x' and t' as a function of x and t using the chain rule:

$$\begin{aligned}\frac{\partial}{\partial x'} &= \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t} = \frac{\partial}{\partial x},\\ \frac{\partial}{\partial t'} &= \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} = v\frac{\partial}{\partial x} + \frac{\partial}{\partial t},\end{aligned}$$

where we have used the transformations we found earlier. Thus, the derivatives in 3D are:

$$\begin{aligned}\boldsymbol{\nabla}' &= \boldsymbol{\nabla},\\ \frac{\partial}{\partial t'} &= \boldsymbol{v}\cdot\boldsymbol{\nabla} + \frac{\partial}{\partial t}.\end{aligned}$$

(b) We define the wave function $\Psi(\boldsymbol{r},t)$ as:

$$\Psi'(\boldsymbol{r}',t') = \Psi(\boldsymbol{r},t)e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}, \tag{5.3}$$

where $\boldsymbol{k}$ and ω are parameters to be determined. By using the result of question (a), show that the wave function $\Psi(\boldsymbol{r},t)$ satisfies the Schrödinger equation in the frame K, provided that the values $\boldsymbol{k}$ and ω are properly chosen.

Since $V'(\boldsymbol{r}',t') = V(\boldsymbol{r},t)$, we can express the Schrödinger equation in terms of the new coordinates using the given expression of Ψ' and the transformation of the derivatives we obtained in the previous question.

For the left-hand side of the Schrödinger equation, we have

$$i\hbar\frac{\partial\Psi'(\boldsymbol{r}',t')}{\partial t'} = i\hbar\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\Psi(\boldsymbol{r},t)e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$$
$$= \left[i\hbar\Big(\boldsymbol{v}\cdot\boldsymbol{\nabla}-i\boldsymbol{v}\cdot\boldsymbol{k}+\frac{\partial}{\partial t}+i\omega\Big)\Psi(\boldsymbol{r},t)\right]e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)},$$

while for the right-hand side,

$$\left(-\frac{\hbar^2}{2m}\boldsymbol{\nabla}'^2+V'(\boldsymbol{r}',t')\right)\Psi'(\boldsymbol{r}',t') = \left[\left(-\frac{\hbar^2}{2m}\Big(\boldsymbol{\nabla}^2-2i\boldsymbol{k}\cdot\boldsymbol{\nabla}-\boldsymbol{k}^2\Big)+V(\boldsymbol{r},t)\right)\Psi(\boldsymbol{r},t)\right]\times$$
$$\times\, e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}.$$

Equating both sides, we get:

$$\left[i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\nabla^2-V(r,t)\right]\Psi(r,t) = i\hbar\left[\frac{\hbar\boldsymbol{k}}{m}-\boldsymbol{v}\right]\cdot\boldsymbol{\nabla}\Psi(\boldsymbol{r},t)+\hbar\left[\omega-\boldsymbol{v}\cdot\boldsymbol{k}+\frac{\hbar\boldsymbol{k}^2}{2m}\right]\Psi(\boldsymbol{r},t)$$

On the LHS of the expression above we have the Schrödinger equation in the frame K. This means that for a valid wave function that satisfies the Schrödinger equation, the RHS must be 0. This is a differential equation for the wave function Ψ. To ensure that it is identically 0 regardless of the wave function, we impose that its coefficients are 0 as well. Thus, we get the following conditions:

$$\hbar\boldsymbol{k} = m\boldsymbol{v}, \quad \hbar\omega = \frac{1}{2}mv^2.$$

(c) The wave functions Ψ' and Ψ, respectively solutions of the Schrödinger equations in K and K', are not the same function. Discuss why this fact does not influence the proof of invariance. Reflect on the meaning of a unit module complex phase which multiplies a wave function.

The principle of (Galilean) invariance imposes that the probabilities of events must be the same when calculated in different inertial reference frames. The wavefunction instead is not directly observable and does not need to be invariant under a Galilean transformation. Here, we see that the wavefunctions $\psi(\mathbf{r},t)$ and $\psi'(\mathbf{r}',t')$ are equal to each other up to a complex number of modulus one, so the probability $P(\mathbf{r},t)$ to find the particle at position $\mathbf{r}$ at time t computed in the frame K will always be equal to the probability $P'(\mathbf{r}',t')$ calculated at the corresponding location $\mathbf{r}' = \mathbf{r}-\mathbf{v}t$.

We can also see that the average value of any operator A diagonal in real space will be the same in the two reference frames. Indeed consider any operator $\hat{A}$ diagonal in real space, such that $\langle x|\hat{A}|\psi\rangle = A(x)\langle x|\psi\rangle = A(x)\psi(x)$. Similarly to what we did for the potential, we assume $A'(r') = A(r)$. We then have:

$$\begin{aligned}\langle\hat{A}\rangle_{K'} &= \int d\boldsymbol{r}' A'(\boldsymbol{r}')|\Psi'(\boldsymbol{r}',t')|^2 \\ &= \int d\boldsymbol{r} A(\boldsymbol{r})\left|\Psi(\boldsymbol{r},t)e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\right|^2 \\ &= \int d\boldsymbol{r} A(\boldsymbol{r})|\Psi(\boldsymbol{r},t)|^2 \\ &= \langle\hat{A}\rangle_K .\end{aligned}$$

The phase factor $e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ has an effect on the averages of the momentum operator. In fact, suppose that the average of the $\hat{p}$ operator in the wavefunction $\Psi'(\mathbf{r}',t')$ is $\mathbf{p}'$. This means that

$$\mathbf{p}'(t') = -i\hbar \int d\mathbf{r}' \Psi'^*(\mathbf{r}',t') \boldsymbol{\nabla}' \Psi(\mathbf{r}',t') \ . \tag{5.4}$$

What is the average momentum in the wavefunction $\Psi(\mathbf{r},t)$ which describes the state in the reference frame K? It is:

$$\begin{aligned}\mathbf{p}(t) &= -i\hbar \int d\mathbf{r}\ \Psi^*(\mathbf{r},t)\boldsymbol{\nabla}\Psi(\mathbf{r},t) \\ &= -i\hbar \int d\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t}\Psi'^*(\mathbf{r}',t')\boldsymbol{\nabla}\left[e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\Psi'(\mathbf{r}',t')\right] \\ &= \hbar\mathbf{k} + \mathbf{p}'(t') \ .\end{aligned} \tag{5.5}$$

(It was used that $\nabla = \nabla'$). Using $\hbar\mathbf{k} = m\mathbf{v}$ we get $\mathbf{p} = \mathbf{p}' + m\mathbf{v}$. This again, agrees with Galilean invariance. An observer in the frame of reference K will see the particle moving at a speed enhanced by $\mathbf{v}$, so the momentum must change by $m\mathbf{v}$.